

An Algebraic Jost-Schroer Theorem for Massive Theories

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Dedicated to the memory of Claudio D'Antoni.

Abstract

We consider a purely massive local relativistic quantum theory specified by a family of von Neumann algebras indexed by the space-time regions. We assume that, affiliated with the algebras associated to wedge regions, there are operators which create only single particle states from the vacuum (so-called polarization-free generators) and are well-behaved under the space-time translations. Strengthening a result of Borchers, Buchholz and Schroer, we show that then the theory is unitarily equivalent to that of a free field for the corresponding particle type. We admit particles with any spin and localization of the charge in space-like cones, thereby covering the case of string-localized covariant quantum fields.

Introduction

The Jost-Schroer theorem states that if the two-point function of a Wightman quantum field is that of a free field, then it coincides with the latter. The theorem has been proved by Schroer [34], Jost [24] and by Federbush and Johnson [20] for the massive case, and by Pohlmeier [32] for the massless case. Steinmann [35] has extended it to quantum fields which are localized on strings (rays) with a fixed space-like direction as explained below.

Here, an analogous theorem is shown in the more general algebraic framework, where the theory is specified by a family of von Neumann algebras indexed by the space-time regions. Such an extension of the theorem is of relevance in view of the existence of “non-local” models which do not correspond to point-localized Wightman fields, as realized in recent years [15–17, 22, 23, 31]. Another relevant aspect of the present article is that its results and methods should be useful in the systematic construction of a quantum field theory for Anyons with “trivial” S-matrix. Of course, both the hypothesis and the conclusion of the theorem need appropriate modifications in the algebraic version. As to the hypothesis, we only consider theories which contain massive particles (of arbitrary spin) separated by a gap from the rest of the mass spectrum. The hypothesis of the Jost-Schroer theorem for Wightman fields is then equivalent to the condition that the fields generate from

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the vacuum only single particle states. So the appropriate hypothesis in the algebraic setting is that there be operators which

1. create only single particle states from the vacuum and are affiliated¹ with certain local algebras, and
2. are well-behaved under the space-time translations in the sense that they give rise to tempered distributions (see Section 1).

In fact, we shall assume only the existence of such operators affiliated with (Rindler) wedge regions². Such operators have been called temperate polarization-free generators (PFG's) by Borchers, Buchholz and Schroer [7]. These authors have shown that if there are temperate PFG's for wedge regions, then the elastic two-particle scattering amplitude vanishes in an open set, which in the case of compact charge localization implies that the S-matrix is trivial. It is well-known [7] that in any purely massive theory there are PFG's (satisfying only the 1st property) for any given wedge region, in fact, there are sufficiently many as to create a dense set in the single-particle space. Thus our hypothesis only concerns the temperate behaviour of these operators under the translations (2nd property). Our conclusion is that then the net of local algebras is unitarily equivalent to that of a free field for the corresponding particle type.³

We admit the most general localization properties for the charge carried by the particles, namely, localization in space-like cones [14].⁴ (For simplicity, we consider only $U(1)$ inner symmetries.) Our result therefore also applies if the local field algebras are generated by string-localized covariant quantum fields as envisaged in [31, 35], inspired by the ideas of Mandelstam [28]: These are operator-valued distributions $\varphi_i(x, e)$ living on the product of Minkowski space and the manifold of space-like directions e , $e \cdot e = -1$. The fields are localized on strings $x + \mathbb{R}_0^+ e$ determined by the pairs (x, e) , namely, $\varphi_i(x, e)$ and $\varphi_j(x', e')$ commute if the corresponding strings are causally separated. Steinmann has shown [35] the Jost-Schroer theorem in the strict sense of Wightman fields for such fields, however only in the special case when the space-like directions of the strings all coincide, namely e is fixed. In contrast, our result covers the case when the fields are actually distributions in e .

The article is organized as follows. In Section 1, we specify in more detail the general setting and the special assumptions, and present the result. In Section 2 we recall some facts on the free field algebras. The remaining two sections contain the proof of the Jost-Schroer theorem: In Section 3 it is shown that the polarization-free generators decompose into a creation and an annihilation part, and that the commutator of two polarization-free generators acts as a multiple of unity. This establishes

¹A closed operator G is said to be affiliated with a von Neumann algebra \mathcal{F} if all elements of the commutant \mathcal{F}' of \mathcal{F} leave its domain invariant and commute with G on its domain.

²A wedge region is any Poincaré transform of the standard wedge W_R which is characterized, in terms of a fixed Lorentz coordinate system, by

$$W_R \doteq \{x \in \mathbb{R}^4 : |x^0| < x^1\}. \quad (1)$$

³It is an interesting open question if the same conclusion also holds without any temperateness assumption but if, instead, (non-temperate) PFG's are assumed to exist also in smaller regions, namely in space-like cones.

⁴A spacelike cone is a region in Minkowski space of the form $C = a + \cup_{\lambda > 0} \lambda \mathcal{O}$, where $a \in \mathbb{R}^4$ is the apex of C and \mathcal{O} is a double cone whose closure is causally separated from the origin.

the Fock space structure. The hermitean polarization-free generators turn out to be self-adjoint, and hence unitarily equivalent to Segal field operators. The remaining problem (Section 4) is to identify the localized single particle vectors $\varphi_0(f)\Omega_0$ of the free theory with single particle vectors created by polarization-free generators. This is accomplished by a single particle version of the algebraic Bisognano-Wichmann theorem [30].

1 Assumptions and Result

We start from a local relativistic quantum theory, specified by a family $C \rightarrow \mathcal{F}(C)$ of von Neumann algebras indexed by the space-time regions C in a certain class. If all charges in the theory are strictly local, the class may be taken to be the double cones, whereas we admit the presence of topological charges [14], hence the class will be taken to be the space-like cones. The algebras $\mathcal{F}(C)$ act in a Hilbert space \mathcal{H} which carries a unitary representation U of the universal covering group \mathcal{P}^c of the Poincaré group with positive energy, *i.e.* the joint spectrum of the generators P_μ of the translations is contained in the closed forward lightcone. There is a unique, up to a factor, invariant vacuum vector Ω . The family $C \rightarrow \mathcal{F}(C)$, together with the representation U , satisfies the following properties.

i) Isotony: $C_1 \subset C_2$ implies $\mathcal{F}(C_1) \subset \mathcal{F}(C_2)$.

ii) Covariance: For all C and all $g \in \mathcal{P}^c$

$$U(g) \mathcal{F}(C) U(g)^{-1} = \mathcal{F}(gC).$$

(To simplify notation, we identify the action of the Poincaré group on Minkowski space with an action of its universal covering group.)

iii) Normal commutation relations: There is a unitary “Bose-Fermi” operator κ , $\kappa^2 = 1$, leaving each field algebra $\mathcal{F}(C)$ invariant, which determines the statistics character of field operators: Field operators which are even/odd under the adjoint action of κ are Bosons/Fermions, respectively. Two field operators which are localized in causally disjoint cones commute if one of the operators is bosonic and anti-commute if both of them are fermionic. This is equivalent to twisted locality [19]: If C_1 and C_2 are spacelike separated, then

$$Z\mathcal{F}(C_1)Z^* \subset \mathcal{F}(C_2)', \quad (2)$$

where Z is the twist operator, $Z \doteq (1 + i\kappa)/(1 + i)$.

iv) Reeh-Schlieder property: For every C , $\mathcal{F}(C)\Omega$ is dense in \mathcal{H} .

We now specify our assumptions. As to the particle content of the theory, we assume that there is one massive particle type (possibly with anti-particle) and no massless ones. We thus make the

Assumption 1 (Massive particle spectrum.) *The mass operator $\sqrt{P^2}$ has one isolated strictly positive eigenvalue m . The corresponding sub-representation $U^{(1)}$ of \mathcal{P}^c is irreducible (neutral case) or has a two-fold degeneracy (charged case).*

We shall call the corresponding eigenspace of the mass operator the single particle space and denote it by $\mathcal{H}^{(1)}$. Our result easily extends to the more general situation of finitely many particle types, and larger inner symmetry groups than $U(1)$. Note that by the spin-statistics theorem [13], the single particle space must be fermionic

(i.e., must be contained in the (-1) -eigenspace of κ) if the spin is half-integer, and bosonic otherwise. Our main assumption, namely the proper hypothesis of the algebraic version of the Jost-Schroer theorem, concerns the polarization-free generators. As mentioned in the introduction, it is well-known [7] that for every wedge W there is a dense set of single particle vectors which are created from the vacuum by polarization-free generators localized in W . Namely, the dense set is the projection onto the single particle space $\mathcal{H}^{(1)}$ of the domain of the Tomita operator $S(W)$ associated with $\mathcal{F}(W)$ and Ω ,⁵ and the polarization-free generator G which creates a given $\psi \in \text{dom} S(W)$ is the closure of the operator

$$G_0 A' \Omega = A' \psi, \quad A' \in \mathcal{F}(W)'. \quad (3)$$

Following [7], we call a polarization-free generator G *temperate* if there is a dense subspace $D(G)$ of its domain, called its domain of temperateness, containing Ω which is invariant under the translations $U(x) \doteq U((x, 1))$, and if for every $\psi \in D(G)$ the function

$$x \mapsto G U(x) \psi$$

is continuous and polynomially bounded in norm for large x , and the same holds for its adjoint G^* . Our main assumption now is that affiliated with every wedge algebra $\mathcal{F}(W)$ there are sufficiently many temperate polarization-free generators as to generate a total subspace from the vacuum. In more detail, we assume the following.

Assumption 2 (Polarization-free generators.) *For each wedge region W there is a self-adjoint⁶ set $\mathcal{G}(W)$ of temperate polarization-free generators affiliated with $\mathcal{F}(W)$. Vectors of the form $G_1 \cdots G_n \Omega$, $G_i \in \mathcal{G}(W_i)$, are well-defined and contained in the domain of temperateness of all polarization-free generators.⁷ For any fixed wedge W , the linear span of vectors of this form with $G_i \in \mathcal{G}(W)$ is dense in \mathcal{H} .*

We shall denote the linear span of vectors of the form $G_1 \cdots G_n \Omega$ with G_i in some $\mathcal{G}(W_i)$, or respectively with all G_i in the same $\mathcal{G}(W)$, by

$$D \quad \text{and} \quad D(W),$$

respectively. Note that set $\mathcal{G}(W)$ is invariant under the one-parameter family $U(\Lambda_W(t))$ representing the boosts $\Lambda_W(t)$ which leave the wedge invariant. For if G is in $\mathcal{G}(W)$, then for any t the operator $U(\Lambda_W(t)) G U(\Lambda_W(-t))$ is also a polarization-free generator affiliated with $\mathcal{F}(W)$, and it is also temperate due to the commutation relations of boosts and translations [6].

The algebraic Jost-Schroer theorem which we are going to prove states that a theory \mathcal{F} satisfying our assumptions is unitarily equivalent to the net \mathcal{F}_0 of a free field for the corresponding particle type (m, s) . By a free field for the particle type at hand (m, s) we mean a Wightman field which generates from the vacuum only single particle states of the corresponding type, that is, which maps the vacuum

⁵We recall the relevant notions on Tomita-Takesaki theory in the Appendix.

The mentioned set is dense due to the commutation relations between the modular objects and the translations established by Borchers [6].

⁶That means, $G \in \mathcal{G}(W)$ implies $G^* \in \mathcal{G}(W)$.

⁷Actually, for our purpose it suffices to consider vectors of the above form where all W_i contain some common space-like cone.

into the representation space $\mathcal{H}^{(1)}$ of the universal covering group of the Poincaré group with mass m and spin s . There are many such fields for a given particle type (m, s) , differing in the representation of $SL(2, \mathbb{C})$ according to which they transform. However, they all generate the same family of von Neumann algebras $W \rightarrow \mathcal{F}_0(W)$ associated with wedge regions up to unitary equivalence, as we recall in Section 2. We show:

Theorem 1 (Jost-Schroer theorem for wedge algebras) *Suppose the theory \mathcal{F} satisfies Assumptions 1 and 2. Then there is a unitary operator V from the Hilbert space \mathcal{H} onto the Fock space over $\mathcal{H}^{(1)}$, such that for any wedge W and any element g in the universal covering group \mathcal{P}^c of the Poincaré group there holds*

$$V \mathcal{F}(W) V^* = \mathcal{F}_0(W), \quad (4)$$

$$V U(g) V^* = U_0(g), \quad (5)$$

$$V \Omega = \Omega_0. \quad (6)$$

Here, U_0 denotes the second quantization of the unitary representation $U^{(1)}$ of \mathcal{P}^c for mass m and spin s given by the restriction of U to $\mathcal{H}^{(1)}$, and Ω_0 denotes the Fock space vacuum.

Now the free field, and consequently the family \mathcal{F} , both satisfy twisted Haag duality for wedge regions (A.2). Therefore their dual nets (A.3) are still local. The dual net \mathcal{F}^d is the maximal local extension of the net \mathcal{F} and has the same physical content as \mathcal{F} . Since \mathcal{F} satisfies twisted Haag duality for wedges, the local dual algebras are given by intersections over wedge algebras,

$$\mathcal{F}^d(\mathcal{O}) = \bigcap_{W \supset \mathcal{O}} \mathcal{F}(W),$$

see Remark A.1 (d). The same considerations hold for the free net. (In fact, $\mathcal{F}_0^d(\mathcal{O})$ coincides with $\mathcal{F}_0(\mathcal{O})$ at least for neutral bosons, see Footnote 9.) Therefore we have

Corollary 2 (Equivalence of the local nets) *The unitary V from the theorem also implements, simultaneously for all double cones \mathcal{O} , the equivalence*

$$V \mathcal{F}^d(\mathcal{O}) V^* = \mathcal{F}_0^d(\mathcal{O}). \quad (7)$$

(In particular, of course, the algebras associated with double cones are non-trivial.)

2 Remarks on the Free Field Nets

A free field for mass m and spin s is a Wightman field $\varphi_0(f)$, $f \in \mathcal{S}(\mathbb{R}^4) \otimes \mathbb{C}^N$, which acts on the (anti-) symmetrized Fock space

$$\mathcal{H}_0 \doteq \Gamma_{\pm}(\mathcal{H}^{(1)}) \quad (8)$$

over the single particle space $\mathcal{H}^{(1)}$ and transforms under some representation of $SL(2, \mathbb{C})$ acting on \mathbb{C}^N . (The Fock space is symmetrized or anti-symmetrized according to whether s is integer or half-integer, respectively.) The field operators $\varphi_0(f)$, $f \in \mathcal{S}(\mathbb{R}^4) \otimes \mathbb{C}^N$, are defined on a common dense domain D_0 , the vectors

with finite particle number. The representation of $SL(2, \mathbb{C})$ under which φ_0 transforms and the property that the field creates from the vacuum only single particle states with mass m and spin s characterize the corresponding free field up to unitary equivalence — this is the content of the Jost-Schroer theorem for Wightman fields. We shall not need an explicit expression for the operators $\varphi_0(f)$ in any of these equivalent representations; what matters here is that they are necessarily of the form $\varphi_0(f) = a^*(\varphi_0(f)\Omega_0) + a(\varphi_0(f)^\dagger\Omega_0)$,⁸ where Ω_0 is the Fock vacuum and $a^*(\phi)$ and $a(\phi)$, $\phi \in \mathcal{H}^{(1)}$, denote the creation and annihilation operators in Fock space with domain D_0 . (The dagger means $A^\dagger \doteq A^*|_{D_0}$.) Therefore the operator $\varphi_0(f) + \varphi_0(f)^\dagger$ coincides on D_0 with the Segal operator

$$G_0(\phi) \doteq a^*(\phi) + a(\phi), \quad (9)$$

where $\phi = (\varphi_0(f) + \varphi_0(f)^\dagger)\Omega_0$, and is essentially self-adjoint on D_0 [33]. The fields $\varphi_0(f)$ with $\text{supp} f$ in a given space-time region \mathcal{O} generate, in some sense, a von Neumann algebra $\mathcal{F}_0(\mathcal{O})$. There are in principle various ways how this can be understood [3, 5, 8], but the minimal requirement is that the closures of $\varphi_0(f) + \varphi_0(f)^\dagger$ with $\text{supp} f \subset \mathcal{O}$ should be affiliated with $\mathcal{F}_0(\mathcal{O})$. Hence the minimal choice is that $\mathcal{F}_0(\mathcal{O})$ be generated by these operators, namely

$$\mathcal{F}_0(\mathcal{O}) = \{e^{i\overline{\varphi_0(f) + \varphi_0(f)^\dagger}} | \text{supp} f \subset \mathcal{O}\}'' . \quad (10)$$

The corresponding net satisfies the Bisognano-Wichmann property and hence twisted Haag duality (A.2) for wedge regions. For wedge regions W the algebras $\mathcal{F}_0(W)$ can therefore not be chosen any larger and (10) is the only possible choice.⁹

We now recall a characterization of $\mathcal{F}_0(W)$ in the context of Tomita-Takesaki theory. Let $S_0(W)$ be the Tomita operator of $\mathcal{F}_0(W)$ and Ω_0 , and let $K_0^{(1)}(W)$ be the intersection of its $+1$ -eigenspace with the single particle space. By construction, the vectors $(\varphi_0(f) + \varphi_0(f)^\dagger)\Omega_0$ with $\text{supp} f$ in W are contained in $K_0^{(1)}(W)$, hence $\mathcal{F}_0(W)$ is contained in the algebra generated by the Segal operators $G_0(\phi)$ with $\phi \in K_0^{(1)}(W)$. But this algebra satisfies the Bisognano-Wichmann property [12]. This has two consequences: Firstly, this apparently larger algebra in fact coincides¹⁰ with $\mathcal{F}_0(W)$:

$$\mathcal{F}_0(W) = \{e^{iG_0(\phi)} | \phi \in K_0^{(1)}(W)\}'' , \quad (11)$$

and secondly, $K_0^{(1)}(W)$ is fixed by the representation $U^{(1)}$ of \mathcal{P}^c up to a unitary which commutes with $U^{(1)}$. Hence the algebra $\mathcal{F}_0(W)$ does not depend on the particular free field which generates it, but only on the particle type, namely on the representation $U^{(1)}$. Note that the same argument shows that $\mathcal{F}_0(W)$ is generated, in the sense of Eq. (11), by any real subspace of $K_0^{(1)}(W)$ which is invariant under the corresponding boosts and has standard closure.

⁸This must be so, again due to the Jost-Schroer theorem, since the two sides of this equation are Wightman fields with the same two-point function satisfying the Klein-Gordon equation.

⁹The same holds for arbitrary contractible regions in the case of neutral free bosonic fields, since in this case Haag duality has been shown [1, 21, 27] for the algebras (10). In the case of charged bosons [18] and of fermions [3, 4, 19], duality has been shown for a net which is defined in a different way than (10) and it is not immediately clear whether the nets coincide.

¹⁰see Remark A.1 (c).

3 Fock Space Structure and Canonical (Anti-) Commutation Relations

By our Assumption 2, for each $f \in \mathcal{S}(\mathbb{R}^4)$ and $G \in \mathcal{G}(W)$, where W is an arbitrary wedge, there exists the strong integral

$$G(f) \doteq \int d^4x f(x) G(x),$$

where $G(x) \doteq U(x)GU(x)^{-1}$. The fact that $G\Omega \in \mathcal{H}^{(1)}$ implies [7, 29] that $G(x)$ is a weak solution of the Klein-Gordon equation $G((\square + m^2)f) = 0$ on the domain D . This gives rise to a unique decomposition,

$$G(f) = G^+(f) + G^-(f), \quad (12)$$

namely the inverse Fourier transforms¹¹ of the distributions $G^+(\cdot)$ and $G^-(\cdot)$ are supported on the positive and negative mass shells H_m^\pm respectively. By standard arguments [25], this implies that for any test function f the operators $G^+(f)$ and $G^-(f)$ have momentum transfer in H_m^+ and H_m^- , respectively. In particular, therefore $G^-(f)$ annihilates the vacuum and maps single particle vectors to multiples of the vacuum vector: If $\phi \in \mathcal{H}^{(1)}$ is in the domain of temperateness of G , then

$$G^-(f)\phi = (\Omega, G^-(f)\phi)\Omega. \quad (13)$$

The action of $G^+(f)$ on single particle vectors has been largely determined by Borchers, Buchholz and Schroer in [7], namely, it maps appropriate single particle vectors onto two-particle scattering vectors. We state their result in a coordinate-free form. The given wedge W defines an order relation¹² \succ_W on the forward light cone

$$V_+ \doteq \{p \in \mathbb{R}^4 \mid p \cdot p > 0, p_0 > 0\},$$

as follows. Let a be some point in the edge, E , of W . Then $E - a$ is a space-like plane which contains the origin. For a given $q \in V_+$, the set $(E - a) + \mathbb{R}q$ is a time-like hyperplane which divides the forward light cone into two connected components. The relation $p \succ_W q$ by definition discriminates the component which extends to space-like infinity in the same direction as W . In formulas: For $p, q \in V_+$ we write $p \succ_W q$ if and only if

$$p \in (W - a) + \mathbb{R}^+q. \quad (14)$$

Given compact sets $V_1, V_2 \subset V_+$, we write $V_1 \succ_W V_2$ if $p \succ_W q$ for every $p \in V_1$ and $q \in V_2$. Now Borchers et al. [7] show that the domain of temperateness of any polarization-free generator contains single particle vectors ϕ with arbitrarily small compact spectral support $\text{sp}_P\phi$, and that for such vectors there holds:

Lemma 3 ([7]) *Suppose that the Fourier transform \hat{f} of f has support in a neighbourhood of a point on the positive mass shell H_m^+ (small enough as to contain no other spectral points). Then*

$$G^+(f)\phi = \begin{cases} (G(f)\Omega \times \phi)_{\text{out}} & \text{if } \text{supp}\hat{f} \succ_W \text{sp}_P\phi \\ (G(f)\Omega \times \phi)_{\text{in}} & \text{if } \text{sp}_P\phi \succ_W \text{supp}\hat{f}. \end{cases} \quad (15)$$

¹¹The Fourier transform \hat{f} of a test function f is given by $\hat{f}(p) \doteq \int d^4x f(x)e^{ip \cdot x}$, and the inverse Fourier transform of the distribution G is given by $\check{G}(\hat{f}) \doteq G(f)$.

¹²That is here a transitive binary relation.

(We show in Appendix B.3 that this is equivalent with Lemma 3.2 in [7].) Moreover, it turns out that the “in”- and “out”-states in Eq. (15) coincide:

Lemma 4 (Triviality of the 2-particle S-matrix [7]) *The incoming and outgoing scattering states constructed from any two single particle states $\psi_1, \psi_2 \in \mathcal{H}^{(1)}$ coincide:*

$$(\psi_1 \times \psi_2)_{\text{in}} = (\psi_1 \times \psi_2)_{\text{out}}. \quad (16)$$

This can be concluded from the work of Borchers et al. [7]. They consider an incoming two-particle state with arbitrary momenta¹³ $q_1 \neq q_2$, and show that there are outgoing momenta p_1, p_2 with $p_1 + p_2 = q_1 + q_2$ such that ${}_{\text{in}}\langle q_1, q_2 | p_1, p_2 \rangle_{\text{out}} = {}_{\text{in}}\langle q_1, q_2 | p_1, p_2 \rangle_{\text{in}}$. (In fact, both sides vanish since their $\{p_1, p_2\}$ are disjoint from $\{q_1, q_2\}$.) The same conclusion is shown to hold if q_1, q_2, p_1, p_2 vary over sufficiently small open sets. In the case of compact localization, it is well-known that this implies the asserted triviality of the 2-particle S-matrix [2, 10]. The proof uses the LSZ relations and momentum space analyticity of both the time-ordered products and of the “intrinsic wave functions” $E^{(1)}B\Omega$. In the case of localization in space-like cones, these analyticity properties are weaker [9], but still sufficient to prove the asserted triviality of the 2-particle S-matrix [11].¹⁴

In the following we consider two wedges W and \hat{W} whose intersection contains some space-like cone C . This situation entails two geometric facts. Firstly, the intersection of their causal complements $W' \cap \hat{W}'$ also contains some spacelike cone \hat{C} , cf. Lemma B.3, hence the algebra $\mathcal{F}(W)' \cap \mathcal{F}(\hat{W})'$ contains a subalgebra for which the vacuum is cyclic, namely $Z\mathcal{F}(\hat{C})Z^*$. Secondly, there are open sets V_1, V_2 in momentum space such that both relations $V_1 \succ_W V_2$ and $V_1 \succ_{\hat{W}} V_2$ hold, cf. Lemma B.5. Let now $G_1 \in \mathcal{G}(W)$ and $G_2 \in \mathcal{G}(\hat{W})$ be two polarization-free generators and let f_1, f_2 be test functions whose Fourier transforms have support in V_1 and V_2 , respectively. Then the Lemmas 3 and 4 imply that

$$G_1^+(f_1)G_2(f_2)\Omega = \varepsilon G_2^+(f_2)G_1(f_1)\Omega. \quad (17)$$

Here, and in the following, $\varepsilon = 1$ in the case of Bosons and $\varepsilon = -1$ in the case of Fermions. (To wit, the l.h.s. of Eq. (17) is $(G_1(f_1)\Omega \times G_2(f_2)\Omega)_{\text{out}}$ and coincides with $\varepsilon(G_2(f_2)\Omega \times G_1(f_1)\Omega)_{\text{in}}$ by the (anti-) symmetry of the scattering states and Lemma 4, which is the r.h.s.) So the difference of the two sides of Eq. (17) is the Fourier transform of a distribution supported on $H_m^+ \times H_m^+$ which vanishes on an open set. We now establish an analyticity property of this Fourier transform (extending Lemma 3.4 in [7]) which implies that it vanishes altogether, that is to say, that Eq. (17) holds for all f_1, f_2 .

In the Fourier decomposition of $G_1 \in \mathcal{G}(W)$ we shall use a Lorentz frame $\{e_0, \dots, e_3\}$ adapted to the wedge W : Let e_2, e_3 be an orthogonal basis of the space-like vector space $E - a$, where E is the edge of W and $a \in E$, and let e_0, e_1 be a pseudo-orthogonal basis of the time-like plane orthogonal to $E - a$ such that \mathbb{R}^+e_1 is contained in $W - a$. We shall denote by x^μ the corresponding contra-variant

¹³Actually they consider q_1, q_2 in the center-of-mass-system of the form $(\omega(\mathbf{q}), \mathbf{q}), (\omega(\mathbf{q}), -\mathbf{q})$, with some condition on the relation of \mathbf{q} with the 1-axes. In view of covariance and our assumption of PFG's for all wedges, these restrictions are obsolete in the present context.

¹⁴I am indebted to Jacques Bros, who has explained to me the argument, as well as the existence of a gap in the case of non-compact localization, and how it should be closed.

components of x , and write $\mathbf{x} \doteq (x^1, x^2, x^3)$, $\mathbf{p} \doteq (p^1, p^2, p^3) \in \mathbb{R}^3$. Since the Fourier transform of the function $G_1(x)$ is supported on the (positive and negative) mass shells, its Fourier decomposition can be written as

$$\begin{aligned} G_1(x) &\equiv \int d^4p \check{G}_1(p) e^{ip \cdot x} \\ &= \int \frac{d^3\mathbf{p}}{2\omega(\mathbf{p})} \{ \Gamma_1^+(\mathbf{p}) e^{i\omega(\mathbf{p})x^0} + \Gamma_1^-(\mathbf{p}) e^{-i\omega(\mathbf{p})x^0} \} e^{-i\mathbf{p} \cdot \mathbf{x}}, \end{aligned} \quad (18)$$

where Γ_1^\pm are (operator valued) distributions on \mathbb{R}^3 and $\omega(\mathbf{p}) \doteq (\mathbf{p}^2 + m^2)^{\frac{1}{2}}$. Moreover, $\Gamma_1^\pm(\mathbf{p})$ has momentum transfer in the positive/negative mass shell, respectively. In the Fourier decomposition of $G_2 \in \mathcal{G}(\hat{W})$ we shall use coordinates \hat{x}^μ w.r.t. a basis \hat{e}_μ adapted to \hat{W} in the analogous way, yielding

$$G_2(y) = \int \frac{d^3\hat{\mathbf{q}}}{2\omega(\hat{\mathbf{q}})} \{ \Gamma_2^+(\hat{\mathbf{q}}) e^{i\omega(\hat{\mathbf{q}})\hat{y}^0} + \Gamma_2^-(\hat{\mathbf{q}}) e^{-i\omega(\hat{\mathbf{q}})\hat{y}^0} \} e^{-i\hat{\mathbf{q}} \cdot \hat{\mathbf{y}}}. \quad (19)$$

In the following we write $\mathbf{p}^\perp \doteq (p^2, p^3)$ and $\hat{\mathbf{q}}^\perp \doteq (\hat{q}^2, \hat{q}^3)$.

Lemma 5 (Momentum space analyticity) *Let W and \hat{W} be wedges as above, let $\{PFG_1 \in \mathcal{G}(W)$ and $G_2 \in \mathcal{G}(\hat{W})$, and let $A \in \mathcal{F}(W)' \cap \mathcal{F}(\hat{W})'$. For \mathbf{p}^\perp and $\hat{\mathbf{q}}^\perp$ in any compact set $M \subset \mathbb{R}^2$ the distribution $(A\Omega, \Gamma_1^+(\mathbf{p})\Gamma_2^+(\hat{\mathbf{q}})\Omega)$ is the boundary value of an analytic function in the variables p^1, \hat{q}^1 , analytic for $p^1, \hat{q}^1 \in (\mathbb{R} - i\mathbb{R}^+) \cap U_M$, where U_M is a complex neighbourhood of the real axis.*

That is to say, for any $f^\perp, g^\perp \in \mathcal{S}(\mathbb{R}^2)$ with supports in M there is a function $(p^1, \hat{q}^1) \mapsto (A\Omega, \Gamma_1^+(p^1, f^\perp)\Gamma_2^+(\hat{q}^1, g^\perp)\Omega)$, analytic in the mentioned strip, such that for every $f, g \in \mathcal{S}(\mathbb{R})$ there holds

$$\begin{aligned} (A\Omega, \Gamma_1^+(f \otimes f^\perp)\Gamma_2^+(g \otimes g^\perp)\Omega) &= \\ \lim_{\varepsilon \rightarrow 0^+} \int dp^1 d\hat{q}^1 f(p^1) g(\hat{q}^1) (A\Omega, \Gamma_1^+(p^1 - i\varepsilon, f^\perp)\Gamma_2^+(\hat{q}^1 - i\varepsilon, g^\perp)\Omega). \end{aligned}$$

Proof. We consider the commutator function

$$K(x, y) \doteq (A\Omega, G_1(x)G_2(y)\Omega) - (G_2^*(y)G_1^*(x)\Omega, A^*\Omega).$$

By the remark before the lemma, the function $G_1(x)G_2(y)\Omega$ can be written as

$$\begin{aligned} G_1(x)G_2(y)\Omega &= \\ \int \frac{d^3\mathbf{p}}{2p_0} \frac{d^3\hat{\mathbf{q}}}{2\hat{q}_0} \{ \phi_+(\mathbf{p}, \hat{\mathbf{q}}) e^{ip_0x^0} + \phi_-(\mathbf{p}, \hat{\mathbf{q}}) e^{-ip_0x^0} \} e^{i\hat{q}_0\hat{y}^0} e^{-i(\mathbf{p} \cdot \mathbf{x} + \hat{\mathbf{q}} \cdot \hat{\mathbf{y}})}, \end{aligned}$$

where $p_0 \doteq \omega(\mathbf{p})$, $\hat{q}_0 \doteq \omega(\hat{\mathbf{q}})$, and ϕ_\pm are the vector-valued distributions $\phi_\pm(\mathbf{p}, \hat{\mathbf{q}}) = \Gamma_1^\pm(\mathbf{p})\Gamma_2^\pm(\hat{\mathbf{q}})\Omega$. Similarly, $G_2^*(y)G_1^*(x)\Omega$ can be represented as

$$\begin{aligned} G_2^*(y)G_1^*(x)\Omega &= \\ \int \frac{d^3\hat{\mathbf{q}}}{2\hat{q}_0} \frac{d^3\mathbf{p}}{2p_0} \{ \psi_+(\hat{\mathbf{q}}, \mathbf{p}) e^{i\hat{q}_0\hat{y}^0} + \psi_-(\hat{\mathbf{q}}, \mathbf{p}) e^{-i\hat{q}_0\hat{y}^0} \} e^{ip_0x^0} e^{-i(\mathbf{p} \cdot \mathbf{x} + \hat{\mathbf{q}} \cdot \hat{\mathbf{y}})} \end{aligned}$$

for some vector-valued distributions ψ_{\pm} . Then the commutator function has the form

$$K(x, y) = \int \frac{d^3 \mathbf{p}}{2p_0} \frac{d^3 \hat{\mathbf{q}}}{2\hat{q}_0} e^{-i(\mathbf{p} \cdot \mathbf{x} + \hat{\mathbf{q}} \cdot \hat{\mathbf{y}})} \times \\ \times \{ K_+(\mathbf{p}, \hat{\mathbf{q}}) e^{i(p_0 x^0 + \hat{q}_0 \hat{y}^0)} + K_-(\mathbf{p}, \hat{\mathbf{q}}) e^{-i(p_0 x^0 + \hat{q}_0 \hat{y}^0)} + K_0(\mathbf{p}, \hat{\mathbf{q}}) e^{-i(p_0 x^0 - \hat{q}_0 \hat{y}^0)} \},$$

where the distributions K_{\pm}, K_0 are given by

$$K_+(\mathbf{p}, \hat{\mathbf{q}}) = (A\Omega, \phi_+(\mathbf{p}, \hat{\mathbf{q}})), \\ K_-(\mathbf{p}, \hat{\mathbf{q}}) = -(\psi_+(-\hat{\mathbf{q}}, -\mathbf{p}), A^*\Omega), \\ K_0(\mathbf{p}, \hat{\mathbf{q}}) = (A\Omega, \phi_-(\mathbf{p}, \hat{\mathbf{q}})) - (\psi_-(-\hat{\mathbf{q}}, -\mathbf{p}), A^*\Omega).$$

Now the restriction of the commutator function to the the time-zero plane $x^0 = 0 = \hat{y}^0$ vanishes if x^1 and \hat{y}^1 are positive, since then A and $G_1(0, \mathbf{x})G_2(0, \hat{\mathbf{y}})$ are localized in causally separated regions. The same holds for its partial derivative with respect to x^0 . Therefore the distributions

$$\frac{1}{\omega(\mathbf{p})\omega(\hat{\mathbf{q}})}(K_+ + K_- + K_0) \quad \text{and} \quad \frac{1}{\omega(\hat{\mathbf{q}})}(K_+ - K_- - K_0)$$

are boundary values (in the sense explained before the proof) of analytic functions in the variables p^1, \hat{q}^1 , analytic in the lower half plane. Furthermore, for \mathbf{p}^\perp in a compact set M , the function $p^1 \mapsto \omega(\mathbf{p})$ is analytic in some neighbourhood U_M of the reals. Hence the distribution $K_+ \equiv (A\Omega, \Gamma_1^+(\mathbf{p})\Gamma_2^+(\hat{\mathbf{q}})\Omega)$ has the analyticity property claimed in the lemma. \square

By Eq. (17), the vector

$$G_1^+(f_1)G_2(f_2)\Omega - \varepsilon G_2^+(f_2)G_1(f_1)\Omega \equiv \\ \int \frac{d^3 \mathbf{p}}{2p_0} \frac{d^3 \hat{\mathbf{q}}}{2\hat{q}_0} \hat{f}_1(p_0, \mathbf{p}) \hat{f}_2(\hat{q}_0, \hat{\mathbf{q}}) \{ \Gamma_1^+(\mathbf{p})\Gamma_2^+(\hat{\mathbf{q}})\Omega - \varepsilon \Gamma_2^+(\hat{\mathbf{q}})\Gamma_1^+(\mathbf{p})\Omega \},$$

$p_0 \doteq \omega(\mathbf{p})$, $\hat{q}_0 \doteq \omega(\hat{\mathbf{q}})$, vanishes if the supports of \hat{f}_1 and \hat{f}_2 are contained in the open sets V_1, V_2 . By the analyticity property established in the last lemma and the fact that $(\mathcal{F}(W)' \cap \mathcal{F}(\hat{W})')\Omega$ is dense (as observed before Eq. (17)), it vanishes altogether. In other words, Eq. (17) holds for any $f_i \in \mathcal{S}(\mathbb{R}^4)$, $G_1 \in \mathcal{G}(W)$ and $G_2 \in \mathcal{G}(\hat{W})$. Together with Eq. (13), this yields

$$[G_1(f_1), G_2(f_2)]_\varepsilon \Omega = c(f_1, f_2) \Omega, \quad (20)$$

where $[A, B]_\varepsilon \doteq AB - \varepsilon BA$ and $c(f_1, f_2)$ is the scalar product of Ω with the left hand side. The same relation holds of course for the non-smeared PFGs, and extends to the subspace D by standard arguments [36, proof of Thm. 4-3]:

$$[G_1, G_2]_\varepsilon = c \, 1 \quad \text{on } D, \quad (21)$$

where c is the vacuum expectation value of the left hand side. This is the “first half” of the Jost-Schroer theorem, and has already been shown in [29] under the assumption that temperate polarization-free generators exist not only for wedges but for space-like cones, and only for PFGs localized in mutually causally separated

space-like cones. (This seems not sufficient for the present purpose where we need the vectors of the form $G_1 \cdots G_n \Omega$ to span a core.) Now recall that $G_1^-(f)$ may be written as $G_1(f_-)$, where $\widehat{f_-}$ results from \widehat{f} by multiplication with a smooth function which is one on the negative and zero on the positive mass shell. Hence

$$[G_1^-(f), G_2]_\varepsilon = c(f) 1$$

holds on D , where $c(f)$ is the vacuum expectation value of the left hand side. Let now W_1, \dots, W_n be wedges whose intersection contains some common space-like cone,

$$\bigcap_{k=1, \dots, n} W_k \supset C, \quad (22)$$

and let $G_k \in \mathcal{G}(W_k)$. Taking into account that $G_1^-(f)$ annihilates the vacuum due to its negative momentum transfer, the above considerations yield

$$G_1^-(f) G_2 \cdots G_n \Omega = \sum_{k=2}^n \varepsilon^{k-1} c_k(f) G_2 \cdots \widehat{G_k} \cdots G_n \Omega \quad (23)$$

for any test function f , where the hat means omission of the corresponding factor, and $c_k(f)$ is the vacuum expectation value of $[G_1^-(f), G_k]_\varepsilon$. To determine this value, note that $\widehat{f_-}$ coincides with \widehat{f} on the positive mass shell, hence $G_1^-(f)^* \Omega = \widehat{f}(P) G_1^* \Omega$ and therefore

$$c_k(f) = (\widehat{f}(P) G_1^* \Omega, G_k \Omega).$$

This shows that the operator-valued distribution $G_1^-(\cdot)$ is, on the sub-space generated by vectors of the form $G_2 \cdots G_n \Omega$, in fact given by a continuous function $G_1^-(x)$ and that $G_1^- \doteq G_1^-(x=0)$ also satisfies Eq. (23) with $c_k(f)$ replaced by $(G_1^* \Omega, G_k \Omega)$. As is well-known from the usual Jost-Schroer theorem, the results obtained so far imply that \mathcal{H} is isomorphic to the (anti-) symmetrized Fock space over $\mathcal{H}^{(1)}$, and that G_1^+ and G_1^- act as creation and annihilation operators, respectively (namely, G_1^+ creates $G_1 \Omega$ and G_1^- annihilates $G_1^* \Omega$.) To set the stage for the next section, we recall the detailed argument.

Equation (23), with $G_1^-(f)$ replaced by G_1^- , implies the recursion relation

$$\begin{aligned} (\Omega, G_1 \cdots G_n \Omega) &= (\Omega, G_1^- G_2 \cdots G_n \Omega) \\ &= \sum_{k=2}^n \varepsilon^k (G_1^* \Omega, G_k \Omega) (\Omega, G_2 \cdots \widehat{G_k} \cdots G_n \Omega) \\ &\equiv (\Omega_0, G_0(\phi_1) \cdots G_0(\phi_n) \Omega_0), \quad \phi_k \doteq G_k \Omega, \end{aligned} \quad (24)$$

where Ω_0 is the Fock space vacuum and $G_0(\phi)$ is the Segal operator (9). The last equation holds because the canonical (anti-) commutation relations imply the same recursion relation for the Segal operators. Now an isomorphism between \mathcal{H} and the (anti-) symmetrized Fock space \mathcal{H}_0 over $\mathcal{H}^{(1)}$ is set up as follows. For any wedge W , let $\mathcal{G}(W)^h$ denote the polarization-free generators $G \in \mathcal{G}(W)$ which satisfy

$$G\Omega = G^*\Omega.$$

These polarization-free generators are hermitean, but in general (i.e., without our Assumption 2) need not be self-adjoint. In any case, there holds

$$G^* = G \quad \text{on } D, \quad (25)$$

since G^* and G both create the same vector from the vacuum and are affiliated with $\mathcal{F}(W)$ (see the proof of Thm. 4-3 in [36] for the argument). For a given wedge W , define an operator V_W from $D(W)$ into \mathcal{H}_0 as follows. For $G_k \in \mathcal{G}(W)^h$, $k = 1, \dots, n$, let

$$V_W G_1 \cdots G_n \Omega \doteq G_0(\phi_1) \cdots G_0(\phi_n) \Omega_0 \quad \text{where } \phi_k \doteq G_k \Omega, \quad (26)$$

$V_W \Omega \doteq \Omega_0$, and extend by linearity to $D(W)$. Eq. (24) implies that V_W is well-defined and isometric. Since $D(W)$ is dense in \mathcal{H} by Assumption 2, V_W extends uniquely to an isometric isomorphism from \mathcal{H} onto \mathcal{H}_0 . Let now \hat{W} be another wedge such that $W \cap \hat{W}$ contains some space-like cone, and let $\hat{G}_1, \dots, \hat{G}_m \in \mathcal{G}(\hat{W})$. Then Eq. (24) implies¹⁵ that

$$\|V_W G_1 \cdots G_n \Omega - V_{\hat{W}} \hat{G}_1 \cdots \hat{G}_m \Omega\| = \|G_1 \cdots G_n \Omega - \hat{G}_1 \cdots \hat{G}_m \Omega\|.$$

Therefore the closures of V_W and $V_{\hat{W}}$ coincide. By iteration of the argument, one sees that the same holds true for any pair of wedges W, \hat{W} . Thus, the closure of $V_{\hat{W}}$, which we shall denote by V_0 , is independent of \hat{W} and satisfies for any wedge W and any $G \in \mathcal{G}(W)^h$ the intertwining relation

$$V_0 G V_0^* = G_0(\phi), \quad \phi = G \Omega, \quad (27)$$

on the dense sub-space $V_0 D(W)$, i.e., on the span of vectors of the form $G_0(\phi_1) \cdots G_0(\phi_n) \Omega_0$, $\phi \in \mathcal{G}(W)$. The restriction of the Segal operator $G_0(\phi)$ to this subspace is hermitean, and all these vectors are analytic for it [33, proof of Thm. X.41]. By Nelson's theorem [33, Thm. X.39], $G_0(\phi)$ is therefore essentially self-adjoint on $V_0 D(W)$. By Eq. (27), the hermitean operator G is essentially self-adjoint on $D(W)$, and the unitary equivalence (27) of course holds for the self-adjoint operators G and $G_0(\phi)$. To summarize, we have shown:

Proposition 6 (PFG's are Segal operators) *Let $G \in \mathcal{G}(W)$ with $G^* \Omega = G \Omega$, where W is any wedge region. Then G is essentially self-adjoint on $D(W)$, and the unitary equivalence (27) holds for the self-adjoint operators G and $G_0(\phi)$, where $\phi \doteq G \Omega$.*

4 Identification of the Local Algebras

We have now identified certain polarization-free generators G with Segal field operators $G_0(\phi)$. However, nothing has been said about the relation of these with the local free field operators $\varphi_0(f)$. We shall clarify this relation for wedge regions and in the context of Tomita-Takesaki theory, whose relevant notions we recall in Appendix A. The unitary equivalence V_0 established in Eq. (27) implements, simultaneously for all wedges W , a unitary equivalence of the von Neumann algebras

$$\hat{\mathcal{F}}(W) \doteq \{e^{iG} \mid G \in \mathcal{G}(W)^h\}'' \subset \mathcal{F}(W) \quad (28)$$

generated by the self-adjoint operators $G \in \mathcal{G}(W)^h$, and the algebras

$$\hat{\mathcal{F}}_0(W) \doteq \{e^{iG_0(\phi)} \mid \phi \in \mathcal{G}(W)^h\}'' \quad (29)$$

¹⁵Note that $W_1 = \cdots = W_n \equiv W$ and $W_{n+1} = \cdots = W_{n+m} \equiv \hat{W}$ satisfy the condition (22) under which Eq. (24) holds.

The relation of $\hat{\mathcal{F}}_0(W)$ with $\mathcal{F}_0(W)$ as defined in Eq. (10) is not immediately clear, and of course we shall not be able to identify a given vector $G\Omega$, $G \in \mathcal{G}(W)^h$, with some $(\varphi_0(f) + \varphi_0(f)^\dagger)\Omega_0$, $\text{supp} f \subset W$.¹⁶ But it suffices to exhibit an isomorphism between the corresponding real subspaces of the single particle space. As mentioned at the end of Section 2, $\mathcal{F}_0(W)$ is generated not only by the set of vectors $(\varphi_0(f) + \varphi_0(f)^\dagger)\Omega_0$ but equally well by any real subspace of $K_0^{(1)}(W)$ with standard closure which is invariant under the boosts corresponding to W . So the crucial point is to identify $\mathcal{G}(W)^h\Omega$ with such a subspace of $K_0^{(1)}(W)$.

Since the operators in $\mathcal{G}(W)^h$ are affiliated with $\mathcal{F}(W)$, the real space $\mathcal{G}(W)^h\Omega$ is obviously contained in the +1-eigenspace

$$K^{(1)}(W)$$

in $\mathcal{H}^{(1)}$ of the Tomita operator of $\mathcal{F}(W)$. Far less obvious is the fact that $K^{(1)}(W)$ and $K_0^{(1)}(W)$ essentially coincide. At this point the algebraic version of the Bisognano-Wichmann theorem [30] comes in: The family of algebras $C \rightarrow \mathcal{F}(C)$ satisfies the hypothesis of [30, Prop. 3] and therefore enjoys the Bisognano-Wichmann property on the single particle space. In particular, the modular unitary group of $\mathcal{F}(W)$ coincides on $\mathcal{H}^{(1)}$ with the representers of the boosts corresponding to the wedge W , and the modular conjugation represents the reflection at the edge of W , see Eq. (A.1). Such operator is unique up to a unitary which commutes with all $U^{(1)}(g)$, $g \in \mathcal{P}^c$. (Of course, in the neutral case it is a multiple of unity, and in the charged case it is a unitary of the degeneracy space \mathbb{C}^2 .) Now the Bisognano-Wichmann property is shared by the free field, and therefore the respective Tomita operators differ by a unitary which commutes with all $U^{(1)}(g)$. We therefore have:

Proposition 7 (Equivalence of the localization structures in $\mathcal{H}^{(1)}$ [30])

There is a unitary operator $V^{(1)}$ which commutes with the representation $U^{(1)}$ of the universal covering group \mathcal{P}^c of the Poincaré group such that for all wedges W there holds

$$V^{(1)}K^{(1)}(W) = K_0^{(1)}(W).$$

Due to this result the real space $V^{(1)}G(W)^h\Omega$ is in $K_0^{(1)}(W)$. It has standard closure and is invariant under the boosts corresponding to W since $\mathcal{G}(W)$ is invariant under these boosts and $V^{(1)}$ commutes with them. Therefore, as mentioned above, the algebra generated by $V^{(1)}G(W)^h\Omega$ coincides with $\mathcal{F}_0(W)$. Summing up, the unitary $V \doteq \Gamma(V^{(1)}) \circ V_0$ implements the equivalence

$$V\hat{\mathcal{F}}(W)V^* = \mathcal{F}_0(W)$$

simultaneously for all wedges W . It remains to show that $\hat{\mathcal{F}}(W)$ is not only a subalgebra but actually coincides with $\mathcal{F}(W)$. To this end, note that the family of free wedge algebras $\mathcal{F}_0(W)$ satisfies twisted Haag duality (A.2), hence the same holds for the family $\hat{\mathcal{F}}(W)$. Then $W \mapsto \mathcal{F}(W)$ is local extension of the Haag

¹⁶This is the crucial difference from the case when \mathcal{F} is generated by a Wightman field $\varphi(f)$ which has the same transformation law under the Poincaré group as the free field $\varphi_0(f)$: In this case, the single particle vectors $\varphi(f)\Omega$ are essentially fixed by covariance and coincide with $\varphi_0(f)\Omega_0$ up to a unitary [37].

dual (and therefore maximal) family of algebras $W \mapsto \hat{\mathcal{F}}(W)$, which implies that $\hat{\mathcal{F}}(W) = \mathcal{F}(W)$. (To wit:

$$\mathcal{F}(W) \subset Z^* \mathcal{F}(W')' Z \subset Z^* \hat{\mathcal{F}}(W')' Z = \hat{\mathcal{F}}(W) \subset \mathcal{F}(W),$$

which implies $\mathcal{F}(W) = \hat{\mathcal{F}}(W)$.) We therefore have shown that V implements the unitary equivalence

$$V \mathcal{F}(W) V^* = \mathcal{F}_0(W) \quad (30)$$

simultaneously for all wedges W . Furthermore, V commutes with the representation of the universal covering group of the Poincaré group, and thus the proof of our theorem is complete.

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A Tomita-Takesaki Theory and the Bisognano-Wichmann Theorem

Let \mathcal{R} be a von Neumann algebra of operators acting in some Hilbert space \mathcal{H} , and let $\Omega \in \mathcal{H}$ be a cyclic and separating vector for \mathcal{R} . The *Tomita operator* associated with \mathcal{R} and Ω is the closed operator S characterized by

$$S A \Omega = A^* \Omega, \quad A \in \mathcal{R}.$$

This is an anti-linear, densely defined, closed operator which is involutive, i.e., satisfies $S^2 \subset 1$. It is therefore completely determined by its $+1$ eigenspace,

$$K \doteq \{ \phi \in \text{dom} S : S \phi = \phi \}.$$

For every vector in the domain of S may be uniquely written as $\psi = \phi_1 + i\phi_2$ with $\phi_1, \phi_2 \in K$, namely $\phi_1 \doteq \frac{1}{2}(\psi + S\psi)$ and $\phi_2 \doteq \frac{1}{2i}(\psi - S\psi)$, and obviously $S(\phi_1 + i\phi_2) = \phi_1 - i\phi_2$ for $\phi_1, \phi_2 \in K$. In particular, K is a real closed subspace of \mathcal{H} which is *standard*, namely $K + iK$ is dense in \mathcal{H} and $K \cap iK = \{0\}$.

Writing the polar decomposition of S as $S = J\Delta^{1/2}$, the operator J is an anti-unitary involution called the modular conjugation, and the positive operator Δ gives rise to the so-called *modular unitary group* Δ^{it} associated with \mathcal{R} and Ω . The Tomita-Takesaki theorem asserts that the adjoint action of the modular unitary group leaves the algebra \mathcal{R} invariant, while that of the modular conjugation maps \mathcal{R} onto its commutant. The following facts are relevant in our context:

Remark A.1 a) Let $\hat{\mathcal{R}}$ be a von Neumann sub-algebra of \mathcal{R} , for which Ω is still cyclic and which is invariant under the adjoint action of the modular unitary group of \mathcal{R} and Ω . Then $\hat{\mathcal{R}} = \mathcal{R}$. (For then $\hat{\mathcal{R}}\Omega$ is dense and invariant under Δ^{it} and therefore is a core for $\Delta^{1/2}$ [33, Thm. VIII.11], which implies that $\hat{\mathcal{R}} = \mathcal{R}$ [26, Thm. 9.2.36].)

b) The *Bisognano-Wichmann theorem* [5,30] states that for a large class of models $\{\mathcal{R}(C)\}_C$ the modular objects J and Δ associated with the algebra $\mathcal{R}(W_R)$ of the standard wedge region W_R and the vacuum Ω are related to the representation of \mathcal{P}^c

under which the fields transform as follows. The modular unitary group coincides with the unitary one-parameter group representing the boosts $\lambda_1(\cdot)$ which leave the wedge W_R invariant, namely $\Delta^{it} = U(\lambda_1(\alpha t))$ for some real constant α . (We call this relation the *Bisognano-Wichmann property*.) Further, the anti-unitary operator J represents the reflection $j = \text{diag}(-1, -1, 1, 1)$ at the edge of the wedge in the sense that $J^2 = 1$ and

$$JU(g)J = U(jgj) \quad (\text{A.1})$$

holds for all $g \in \mathcal{P}^c$. As a consequence of these relations, the field satisfies *twisted Haag duality* for wedge regions, i.e.,

$$Z \mathcal{R}(W) Z^* = \mathcal{R}(W')'. \quad (\text{A.2})$$

The Bisognano-Wichmann theorem holds for the free fields, and has been shown to hold in the algebraic setting for any theory satisfying our Assumption 1 and asymptotic completeness [30].

c) Let $\{\mathcal{R}(C)\}_C$ be a model which satisfies the Bisognano-Wichmann property. Suppose that for some wedge W , $\mathcal{R}_0(W)$ is a von Neumann subalgebra of $\mathcal{R}(W)$ for which the vacuum is still cyclic and which is invariant under the representers of the boosts leaving W invariant. Then $\mathcal{R}_0(W)$ coincides with $\mathcal{R}(W)$ by part (a) of the remark.

d) Twisted Haag duality for wedges implies that the so-called dual net, defined by

$$\mathcal{R}^d(\mathcal{O}) \doteq (Z^* \mathcal{R}(\mathcal{O}') Z) ', \quad (\text{A.3})$$

is still local and that it coincides for convex causally complete regions \mathcal{O} with

$$\mathcal{R}^d(\mathcal{O}) = \bigcap_{W \supset \mathcal{O}} \mathcal{R}(W), \quad (\text{A.4})$$

where the intersection goes over all wedge regions containing \mathcal{O} . (The proof goes as that of Cor. 3.5 in [12].) It is the maximal local extension of the net $\mathcal{R}(\mathcal{O})$ in the sense that it contains every local extension of it, and has the same physical content, for example the same S-matrix.

B Geometric Considerations

B.1 Intersections of Wedges.

In section 3 we used some rather obvious properties of the class of space-like cones and wedges which we prove here for completeness' sake.

Lemma B.1 *Let C be a space-like cone with apex b , and let W be a wedge with apex a . (That is, a is contained in the edge of W .) Then $C \subset W$ if and only if $b \in \overline{W}$ and $C - b \subset W - a$.*

Proof. Since both C and W are convex, $C \subset W$ is equivalent with $\overline{C} \subset \overline{W}$. Now $\overline{C} = b + \mathbb{R}_0^+ \overline{\mathcal{O}}$ where $\overline{\mathcal{O}}$ is causally separated from the origin. Let $e \in \overline{\mathcal{O}}$. According to Lemma A.1 of [30], $b + \mathbb{R}_0^+ e \subset W$ is equivalent with $b \in W$ and $e \in \overline{W} - a$. With the same method one shows that $b + \mathbb{R}_0^+ e \subset \overline{W}$ if and only if $b \in \overline{W}$ and $e \in \overline{W} - a$. Hence $\overline{C} \subset \overline{W}$ is equivalent with $b \in \overline{W}$ and $\overline{\mathcal{O}} \subset \overline{W} - a$. But $\overline{C} \subset \overline{W} \Leftrightarrow C \subset W$, and $\overline{\mathcal{O}} \subset \overline{W} - a \Leftrightarrow C - b \subset W - a$, which proves the claim. \square

Lemma B.2 *Let W_0 and \hat{W}_0 be wedges whose edges contain the origin, and let C_0 be a space-like cone with apex at the origin such that $C_0 \subset W_0 \cap \hat{W}_0$. Then for every $a, \hat{a} \in \mathbb{R}^4$ there exists some $b \in \mathbb{R}^4$ such that*

$$C_0 + b \subset (W_0 + a) \cap (\hat{W}_0 + \hat{a}). \quad (\text{B.1})$$

Proof. By Lemma B.1, it is sufficient to show that the closures of $W_0 + a$ and $\hat{W}_0 + \hat{a}$ have non-empty intersection. As a first step, note that for every $e \in W_0$ and $\tilde{a} \in \mathbb{R}^4$ there holds $\lambda e \in W_0 + \tilde{a}$ for all sufficiently large $\lambda \in \mathbb{R}$. (This can be verified taking W_0 the standard wedge. Then $\lambda e \in W_0 + \tilde{a}$ if and only if $\lambda e^1 > \tilde{a}^1$ and $((e^1)^2 - (e^0)^2)\lambda^2 + \beta\lambda + \gamma > 0$, where β and γ are some real numbers. Since $e^1 > 0$ and $(e^1)^2 - (e^0)^2 > 0$ by hypothesis, both conditions are satisfied for sufficiently large λ .) This result implies that for every $e \in W_0 \cap \hat{W}_0$ (which is non-empty by hypothesis) there is some $\lambda \in \mathbb{R}^+$ such that

$$\lambda e \in (W_0 + a - \hat{a}) \cap \hat{W}_0.$$

Then $b := \lambda e + \hat{a}$ is contained in $(W_0 + a) \cap (\hat{W}_0 + \hat{a})$, completing the proof. \square

Lemma B.3 *If the intersection of two wedges $W \cap \hat{W}$ contains some space-like cone C , then the intersection of their causal complements $W' \cap \hat{W}'$ contains the space-like cone $-C + c$ for some appropriately chosen $c \in \mathbb{R}^4$.*

Proof. By Lemma B.1, the hypothesis implies that $(W - a) \cap (\hat{W} - \hat{a})$ contains $C - b$, that is, $-C + b \subset (-W + a) \cap (-\hat{W} + \hat{a})$. Here a, \hat{a} are in the respective edges of W, \hat{W} and b is the apex of C . Now the cone and wedges involved have apex and edges at the origin. Hence Lemma B.2 applies, to the effect that there is some \hat{b} such that $-C + b + \hat{b} \subset (-W + 2a) \cap (-\hat{W} + 2\hat{a})$. But $-W + 2a$ is just W' , and similarly for \hat{W}' , and thus the proof is complete. \square

B.2 The Order Relation on Velocity space.

Our definition (14) of the order relation $p \succ_W q$ induced by a wedge W is obviously equivalent with

$$p \in (E - a) + \mathbb{R}^+ e + \mathbb{R}^+ q, \quad (\text{B.2})$$

where E is the edge of W , $a \in E$, and where e is any fixed vector in the interior of $W - a$. In this form, one sees easily that the relation \succ_W is transitive. Further, relation (B.2) is obviously equivalent with $q \in (E - a) - \mathbb{R}^+ e + \mathbb{R}^+ p$, hence with

$$q \in -(W - a) + \mathbb{R}^+ p. \quad (\text{B.3})$$

We therefore have

Lemma B.4 *The relation \succ_W defined on V_+ by Eq. (14) is transitive. The relation $p \succ_W q$ is equivalent with $q \succ_{-W} p$.*

In Section 3 we have used the following fact.

Lemma B.5 *Let W and \hat{W} be wedges whose intersection contains some space-like cone C . Then there are open subsets V_1, V_2 of V_+ which have non-trivial intersection with H_m^+ and satisfy both $V_1 \succ_W V_2$ and $V_1 \succ_{\hat{W}} V_2$.*

Proof. Let a and \hat{a} be points in the edges of W and \hat{W} respectively. By Lemma B.1, $(W - a) \cap (\hat{W} - \hat{a})$ contains the cone $C - b =: C_0$, where b is the apex of C . Fix some velocity vector $k \in V_+$, and define

$$V_1 \doteq (C_0 + \mathbb{R}k) \cap V_+, \quad V_2 \doteq (-C_0 + \mathbb{R}k) \cap V_+.$$

Then $V_1 \succ_W k$ and $V_1 \succ_{\hat{W}} k$. Further, by Lemma B.4 or Eq. (B.3), $k \succ_W V_2$ and $k \succ_{\hat{W}} V_2$. By transitivity (Lemma B.4), this proves the claim. \square

For the proof that our order relation is the covariant generalization of the order relation used in [7], we shall need:

Lemma B.6 *For the standard wedge W_R , the relation $p \succ_{W_R} q$ is equivalent with*

$$\frac{p^1}{p^0} > \frac{q^1}{q^0}. \quad (\text{B.4})$$

Here, p^μ denote the contra-variant components of p with respect to the Lorentz frame $\{e_0, \dots, e_3\}$ adapted to W_R (i.e., the unit vectors e_2 and e_3 span the edge of W_R and the ray $\mathbb{R}^+ e_1$ is contained in W_R).

Proof. We first show that

$$\frac{p^1}{p^0} = \frac{q^1}{q^0} \Leftrightarrow p \in E + \mathbb{R}q, \quad (\text{B.5})$$

where E is the edge of W_R . The direction “ \Leftarrow ” is clear, and we show “ \Rightarrow ”. Let \bar{e}_0, \bar{e}_1 be any pseudo-orthogonal basis of the orthogonal complement E^\perp such that \bar{e}_0 is time-like future pointing and \bar{e}_1 points into the same direction as e_1 , that is, $\mathbb{R}^+ \bar{e}_1 \subset W_R$. By application of a boost one can transform the basis $\{e_0, e_1\}$ of E^\perp into $\{\bar{e}_0, \bar{e}_1\}$. But the condition $\frac{p^1}{p^0} = \frac{q^1}{q^0}$ is invariant under these boosts, and hence equivalent with $\frac{\bar{p}^1}{\bar{p}^0} = \frac{\bar{q}^1}{\bar{q}^0}$, where \bar{p}^μ are the (contra-variant) coordinates of p with respect to $\{\bar{e}_0, \bar{e}_1\}$. (For

$$\frac{(\Lambda_1(t)p)^1}{(\Lambda_1(t)p)^0} = \frac{p^1 + \tanh(t)p^0}{p^0 + \tanh(t)p^1}$$

and $\frac{p^1 + cp^0}{p^0 + cp^1} = \frac{q^1 + cq^0}{q^0 + cq^1}$ if and only if $\frac{p^1}{p^0} = \frac{q^1}{q^0}$.) Now let, specifically, \bar{e}_0, \bar{e}_1 be the pseudo-orthogonal basis of the orthogonal complement E^\perp such that \bar{e}_0 is the normalized projection of q onto E^\perp . Then q lies in the linear span of E and \bar{e}_0 , hence $\bar{q}^1 = 0$. Then the condition $\frac{\bar{p}^1}{\bar{p}^0} = \frac{\bar{q}^1}{\bar{q}^0}$ implies that p also lies in the linear span of E and \bar{e}_0 , which coincides with $E + \mathbb{R}q$. This proves the equivalence (B.5).

Now $E + \mathbb{R}q$ is a time-like hyperplane which divides V_+ into two connected components. The relation $\frac{p^1}{p^0} > \frac{q^1}{q^0}$ discriminates (for fixed q) those p contained the component which extends to space-like infinity in the direction e_1 . But the same component is discriminated by the relation $p \succ_{W_R} q$. Therefore the relations coincide. \square

B.3 Proof of Lemma 3.

We now show that our Lemma 3 is equivalent with Lemma 3.2 of Borchers et al. [7].

Suppose that the Fourier transform \hat{f} of f has support in a neighbourhood of a point on the positive mass shell H_m^+ (small enough as to contain no other spectral points). Borchers et al. [7] define the velocity support $V(f)$ of f , with respect to some fixed time-like unit vector as reference frame, by

$$V(f) \doteq \left\{ \left(1, \frac{\mathbf{p}}{\omega(\mathbf{p})} \right), p = (p_0, \mathbf{p}) \in \text{supp} \hat{f} \right\}, \quad (\text{B.6})$$

where $\omega(\mathbf{p}) \doteq (\mathbf{p}^2 + m^2)^{\frac{1}{2}}$. The velocity support $V(\phi)$ of a single particle vector $\phi \in \mathcal{H}^{(1)}$ is defined analogously, with $\text{supp} \hat{f}$ replaced by the spectral support $\text{sp}_P \phi$ of ϕ . Borchers et al. [7] consider the standard wedge W_R and define a partial order relation $>$ on velocity space, namely, given compact sets $V_1, V_2 \subset \mathbb{R}^4$, they write $V_1 > V_2$ if the set of difference vectors $V_1 - V_2$ is contained in W_R . They then show that there holds¹⁷

$$G^+(f)\phi = \begin{cases} (G(f)\Omega \times \phi)_{\text{out}} & \text{if } V(f) > V(\phi) \\ (G(f)\Omega \times \phi)_{\text{in}} & \text{if } V(\phi) > V(f). \end{cases} \quad (\text{B.7})$$

Let us first reformulate the condition on the velocity supports in (B.7). For a momentum vector $p \in V_+$ define the corresponding velocity $v(p) := (1, \frac{\mathbf{p}}{\omega(\mathbf{p})})$, and for two momentum vectors p, q define $p >_{W_R} q$ if and only if

$$v(p) - v(q) \in W_R. \quad (\text{B.8})$$

Then $V(f) > V(\phi)$ if and only if $p >_{W_R} q$ for all $p \in \text{supp} \hat{f}$ and $q \in \text{sp}_P \phi$. Since the support of the Fourier transform of G^+ is contained in the positive mass shell H_m^+ , one may of course replace $\text{supp} \hat{f}$ with $\text{supp} \hat{f} \cap H_m^+$ in the definition (B.6) of $V(f)$, and the conclusion (B.7) still holds. Now for an arbitrary wedge $W = \Lambda W_R + a$ let us define $p >_W q$ if and only if

$$\Lambda^{-1}p >_{W_R} \Lambda^{-1}q. \quad (\text{B.9})$$

By covariance, and using $U(g)^{-1}G(f)U(g) = U(g)^{-1}GU(g)(g^*f)$, where g^* is the pull-back action of $g \in \mathcal{P}^c$, one readily verifies that the result (B.7) of Borchers et al. is equivalent with our Lemma 3 — with the relation $>_W$ instead of the relation \succ_W defined in Eq. (14). It remains to show that these two relations coincide on the mass shell.

Lemma B.7 *On the mass shell, the relation \succ_W coincides with the relation $>_W$ defined in (B.8), (B.9).*

Proof. In a first step, we consider the standard wedge W_R . For a point p on the mass shell, $v(p)$ is just p/p^0 . Hence for $p, q \in H_m^+$ the relation $p >_{W_R} q$ is equivalent with $\frac{p^1}{p^0} > \frac{q^1}{q^0}$, and hence with $p \succ_{W_R} q$ by Lemma B.6. Let now $W = \Lambda W_R + a$

¹⁷This is the content of Lemma 3.2 in [7]. The lemma treats only the case of Bosons with compact localization explicitly but, as the authors assert, easily extends to the case of Fermions and of localization in space-like cones.

be an arbitrary wedge. Recall that $p >_W q$ means by definition $\Lambda^{-1}p >_{W_R} \Lambda^{-1}q$, which we have just shown to be equivalent with $\Lambda^{-1}p \in W_R + \mathbb{R}\Lambda^{-1}q$, that is, with $p \in \Lambda W_R + \mathbb{R}q$. But ΛW_R is just the translated wedge $W - a$, hence $p >_W q$ is equivalent with $p \succ_W q$. \square

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